The $1/N$ expansion of colored tensor models in arbitrary dimension

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In this paper we extend the $1/N$ expansion introduced in [1] to group field theories in arbitrary dimension and prove that only graphs corresponding to spheres S^D contribute to the leading order in the large N limit.

INTRODUCTION

The $1/N$ expansion, together with Wilson's-Fisher $\epsilon = 4 - D$ expansion around mean field theory and the $1/D$ expansion of condensed matter around dynamical mean field theory, stands out as the main alternative to the ordinary coupling constant expansion or to numerical simulations in quantum field theory and statistical mechanics. The (large) parameter N is a range of integer values for fields indices encoding some tensorial structure. Until now the $1/N$ expansion was understood solely for vector or matrix models but not for higher rank tensor fields.

The leading terms in this expansion for the free energy are linear chains (rings) made of a simple bubble motive for vector models [2] and planar graphs for matrix models [3, 4]. The corresponding Feynman graphs obviously pave respectively the circle S^1 and the 2 dimensional sphere S^2. It is therefore tempting to conjecture that higher rank tensor fields with D indices should lead to models which admit also a $1/N$ expansion in which dominant graphs pave the D dimensional sphere S^D. However no examples of such a higher rank $1/N$ expansion had been found until recently.

In [1] the first expansion of this type was found for the *colored* [5] Boulatov model [6] (which is a group field theory in $D = 3$). The color condition seems a key ingredient for this $1/N$ expansion to hold, or at least to be tractable. Its graphs are duals to manifolds with only point-like singularities [7]. Remarkably, the colors are also necessary to implement the diffeomorphism symmetry in GFT [8]. The main reason for the interest in group field theory lies in its potential for quantizing gravity [9, 10]. Just as matrix models are related to 2-D gravity [11, 12], it is expected that D-rank group field models are related to D-dimensional gravity [13, 14]. The physically interesting case is then $D = 4$, hence the $1/N$ expansion for rank-four tensors is even more important than the one for rank-three tensors.

Ideally we would like quantum gravity to be based on a simple quantum field theory, since this is the case for all other known forces of physics. But we would like also to understand why space-time, which is expected to be wild and foamy at the Planck scale or beyond, is so smooth and manifold-like at large distances. Some mechanism should flatten out the virtual loops and handles of quantum space-time and favor the trivial S^4 topology, since this is the long-distance classical world that we actually observe.

In this letter we extend the method of [1] to prove that the *colored* group field theory in D dimensions indeed admits a $1/N$ expansion dominated by graphs which pave the S^D manifold. This includes the physically interesting case of $D = 4$, namely the colored topological Ooguri model. Realistic models for quantum gravity should include dynamical degrees of freedom, as is attempted eg. in the EPR-FK models [15–18], and we hope some more complicated form of the $1/N$ expansion can be extended also to the colored version of these models.

We would like to stress that, to our knowledge, the results in this paper give the first analytic example of such a dominance of S^4 topologies in a quantum field theory model. Dominance of large and smooth structures is indeed not easy to obtain in sums over random space-times: most naive models tend to develop crumpled or polymer-like phases. To our knowledge the only other approach in which simulations of random space-times lead to large and smooth structures is causal dynamical triangulations [19], but they are based exclusively on numerical and not analytic results.

Our result is the following. We consider the D-dimensional GFT theory introduced in [5]. The corresponding $D + 1$ colors are noted as 0,1 ... D, and the vertices are noted sloppily as $\lambda\phi^{D+1}$ and $\bar\lambda(\bar\phi)^{D+1}$. Introducing a cutoff on the group representations provides the index N (i.e. we suppress representations of spin higher than N). The corresponding regularized δ function on the group diverges at the origin when $N \to \infty$ as $\delta^N(e)$. We prove below that the free energy $F = \log Z$ of the theory obeys

$$F = \delta^N(e)^{D-1} C(\lambda, \bar\lambda) + O[\delta^N(e)^{D-1-\frac{2(D-2)}{D!}}] \,, \quad (1)$$

where all graphs contributing to $C(\lambda, \bar\lambda)$ have the topology of spheres S^D.

We *do not* prove that the subleading terms in (1) are also indexed by topologies. We explain why in $D = 3$ it has been possible to go further and prove that the entire series is index by topologies (encoded in the *core graphs* of [1]). We expect an analog of this to be also true in arbitrary D, but establishing this is beyond the scope of this paper.

GRAPHS, JACKETS AND BUBBLES

Let G be some compact multiplicative Lie group, and denote h its elements, e its unit, and $\int dh$ the integral with respect to the Haar measure. Let $\bar\psi^i, \psi^i$, $i = 0, 1, \dots, D$ be $D+1$ couples of complex scalar (or Grassmann) fields over D copies of G, $\psi^i : G^{\times D} \to \mathbb{C}$. Denote $\psi(h_0, \dots, h_{D-1}) := \psi_{h_0,\dots,h_{D-1}}$. The partition function of the D dimensional colored GFT [1, 5, 7] is defined by the path integral

$$e^{-F} = Z(\lambda, \bar\lambda) = \int \prod_{i=0}^{D} d\mu_P(\psi^i, \bar\psi^i)\; e^{-S^{int} - \bar S^{int}} , \quad (2)$$

with normalized Gaussian measure of covariance P and interaction S^{int}

$$\begin{aligned} P_{h_0\dots h_{D-1}; h'_0 \dots h'_{D-1}} &= \int dh \prod_{i=0}^{D-1} \delta^N\left(h_i h (h'_i)^{-1}\right) , \\ S^{int} &= \frac{\lambda}{\sqrt{\delta^N(e)^{\frac{(D-2)(D-1)}{2}}}} \int \prod_{i<j} dh_{ij} \\ & \prod_{i=0}^{D+1} \psi^i_{h_{ii-1}\dots h_{i0} h_{in} \dots h_{ii+1}} , \end{aligned} \quad (3)$$

where $h_{ij} = h_{ji}$. The index $i \in \{0, \dots, D\}$ of each field is a *color* index. The Feynman graphs of the D-dimensional colored GFT (called (D+1)- colored graphs), are made of oriented colored lines with D parallel threads and oriented vertices (dual to D simplices) of coordination $D+1$. For every vertex, the thread (i, j) connects the halflines of colors i and j (see figure 1). The 3-colored graphs are the familiar ribbon graphs of matrix models.

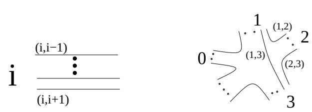

FIG. 1. Line and vertex of the Colored GFT graphs.

Colored graphs [5, 7] are dual to normal pseudo manifolds. The free energy F of the colored GFT writes as a sum over connected vacuum colored graphs $\mathcal{G}$. We denote $\mathcal{N}_\mathcal{G}$, $|\mathcal{N}_\mathcal{G}| = 2p$, $\mathcal{L}_\mathcal{G}$, $\mathcal{F}_\mathcal{G}$, the sets of vertices, lines and faces (i.e. closed threads) of $\mathcal{G}$. The amplitude of $\mathcal{G}$, $A^\mathcal{G}$ is [1, 5, 20]

$$\frac{(\lambda\bar\lambda)^p}{[\delta^N(e)]^{p\frac{(D-2)(D-1)}{2}}} \int \prod_{\ell \in \mathcal{L}_\mathcal{G}} dh_\ell \prod_{f \in \mathcal{F}_\mathcal{G}} \delta^N_f\Big(\overrightarrow{\prod_{\ell \in f}} h_\ell^{\sigma^{\ell|f}}\Big) , \quad (4)$$

where $\sigma^{\ell|f} = 1$ (resp. -1) if the orientations of ℓ and f coincide (resp. are opposite) and $\sigma^{\ell|f} = 0$ if ℓ does not belong to the face f.

Consider the two point colored graph with two vertices connected by D lines (denoted $\mathcal{G}_1$) represented in figure

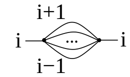

FIG. 2. The two point graph $\mathcal{G}_1$.

2. For any graph $\mathcal{G}$, one can consider the family obtained by inserting $\mathcal{G}_1$ an arbitrary number of times on any line of $\mathcal{G}$. The scaling of the coupling constants in eq. (3) is the only scaling which ensures that this family has uniform degree of divergence, hence it is the only scaling under which a $1/N$ expansion makes sense.

To a colored graph $\mathcal{G}$ one associates two categories of subgraphs: its bubbles [5, 7] and its jackets [1, 20]. The 0-bubbles of $\mathcal{G}$ are its vertices and the 1-bubbles are its lines. For $p \ge 2$, the *p-bubbles* with colors $\{i_1, \dots, i_p\}$ of $\mathcal{G}$ are the connected components (labeled ρ) obtained from $\mathcal{G}$ by deleting the lines and faces containing at least one of the colors $\{0, \dots, D\} \setminus \{i_1, \dots, i_p\}$. We denoted the p-bubbles $\mathcal{B}^{i_1\dots i_p}_{(\rho)}$. For $p \ge 2$ each p-bubble is a p-colored graph, and the 2-bubbles are the faces of $\mathcal{G}$.

We denote $\widehat{i} = \{0, \dots, D\} \setminus \{i\}$. Consider a line (say of color 0) l^0 with end vertices v and w in a graph $\mathcal{G}$. Each of the vertices v and w belongs to some D-bubble of colors $\widehat{0}$, $\mathcal{B}^{\widehat{0}}_{(\alpha)}$ and $\mathcal{B}^{\widehat{0}}_{(\beta)}$. If the two bubbles are *different* and at least one of them is dual to a *sphere* S^{D-1}, then l^0 is a **1-Dipole** [1, 21, 22]. A 1-Dipole can be contracted, that is the line l^0 together with the vertices v and w can be deleted from the graph and the remaining lines reconnected *respecting the coloring* (see figure 3). The fundamental result we will use in the sequel [21] is that the two pseudo manifolds dual to $\mathcal{G}$ and $\mathcal{G}'$ are homeomorphic if $\mathcal{G}$ and $\mathcal{G}'$ are related by a 1-Dipole contraction. We call two such graphs "equivalent", $\mathcal{G} \sim \mathcal{G}'$.

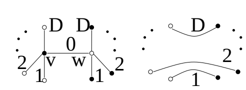

FIG. 3. Contraction of a 1-Dipole.

Definition 1. *A colored* **jacket** *$\mathcal{J}$ of $\mathcal{G}$ is the ribbon graph made by the faces of colors $(\tau^k(0), \tau^{k+1}(0))$, for some cycle τ of $D+1$ elements, modulo the orientation of the cycle. We denote $J = \{(\tau^k(0), \tau^{k+1}(0))\,|k = 0, \dots, D\}$ the set of faces of $\mathcal{J}$.*

For $D = 2$ the (unique) jacket of a colored graph is the graph itself. The reader can check that $\mathcal{J}$ and $\mathcal{G}$ have the same connectivity, the number of jackets of a $D+1$ colored graph is $\frac{1}{2}D!$ and the number of jackets containing a given pair is $(D-1)!$. In $D = 4$ there are 5 colors, 10 pairs of colors and 12 jackets (corresp. to the cycles (01234), (01243), etc.). The ribbon lines of the jacket $\mathcal{J}$ separate two faces, $(\tau^{-1}(j), j)$ and $(j, \tau(j))$ and

inherit the color j of the line in $\mathcal{G}$. As the p-bubbles are p-colored graphs they also possess jackets which can be obtained from the jackets of $\mathcal{G}$.

For a jacket $\mathcal{J}$, $J^{\widehat{i}} = J \setminus \{(\tau^{-1}(i), i), (i, \tau(i))\} \cup \{(\tau^{-1}(i), \tau(i))\}$ is a cycle over the D elements $\widehat{i}$. The ribbon subgraph of $\mathcal{G}$ made of faces in $J^{\widehat{i}}$ is the union of several connected components, $\mathcal{J}^{\widehat{i}}_{(\rho)}$. Each $\mathcal{J}^{\widehat{i}}_{(\rho)}$ is a jacket of the D-bubble $\mathcal{B}^{\widehat{i}}_{(\rho)}$. Conversely, every jacket of $\mathcal{B}^{\widehat{i}}_{(\rho)}$ is obtained from exactly D jackets of $\mathcal{G}$ corresponding to inserting the color i anywhere along the cycle of D elements.

Lemma 1. *If a jacket $\mathcal{J}$ is planar then, for all i and ρ, the jacket graphs $\mathcal{J}^{\widehat{i}}_{(\rho)}$ of the D-bubbles are planar.*

Proof: We delete one by one all lines of color i. We denote $\mathcal{J} \setminus l^i$ the graph obtained from $\mathcal{J}$ by deleting the line l^i. As long as the two faces of $\mathcal{J} \setminus l^i_1 \cdots \setminus l^i_{p-1}$ touching l^i_p are different, the number of faces decreases by 1 when deleting l^i_p, hence the Euler character χ of $\mathcal{J} \setminus l^i_1 \cdots \setminus l^i_{p-1}$ equals the Euler character χ' of $\mathcal{J} \setminus l^i_1 \cdots \setminus l^i_{p-1} \setminus l^i_p$.

Suppose that we reach a line l^i_p such that the two sides of the ribbon belong to the same face of $\mathcal{J} \setminus l^i_1 \cdots \setminus l^i_{p-1}$. Erasing l^i_p increases the number of faces by one, hence $\chi' = \chi + 2$. As the Euler character factors over connected components, we conclude that erasing l^i_p necessarily divides some (planar) connected component of the $\mathcal{J} \setminus l^i_1 \cdots \setminus l^i_{p-1}$ into two planar connected components of $\mathcal{J} \setminus l^i_1 \cdots \setminus l^i_{p-1} \setminus l^i_p$. Hence all $\mathcal{J}^{\widehat{i}}_{(\rho)}$ are planar.

□

THE $1/N$ EXPANSION

All the group elements of the lines in a tree $\mathcal{T} \in \mathcal{G}$ can be eliminated from the amplitude (4) by a tree change of variables [23]. Consider a jacket $\mathcal{H}$ of $\mathcal{G}$. The lines of $\mathcal{H}$ admit (many) partitions in three disjoint sets: a tree $\mathcal{T}$ in $\mathcal{H}$, a tree $\tilde{\mathcal{T}}$ in the dual graph $\tilde{\mathcal{H}}$, and a set $\mathcal{L} \setminus \mathcal{T} \setminus \tilde{\mathcal{T}}$, of "genus" lines [24]. Denoting $l(f, \tilde{\mathcal{T}})$ the line in the dual tree $\tilde{\mathcal{T}}$ touching the face f and going towards the root face r, the contribution of the faces of $\mathcal{H}$ to the amplitude (4) can be cast into the form [1]

$$\prod_{f \in \mathcal{H}} \delta^N_f \Big(\prod_{\ell}^{\rightarrow} h_\ell^{\sigma^{\ell|f}}\Big) = \delta^N_r \Big(\prod_{\ell \notin \tilde{\mathcal{T}}}^{\rightarrow} h_\ell^{\sigma^{\ell|\cup_{f \in \mathcal{H}} f}}\Big) \times \prod_{f \in \mathcal{H}, f \neq r} \delta^N_f \Big(h_{l(f,\tilde{\mathcal{T}})}^{\sigma^{l(f,\tilde{\mathcal{T}})|f}} \big(\prod_{\ell \neq l(f,\tilde{\mathcal{T}})}^{\rightarrow} h_\ell^{\sigma^{\ell|f}} \big) \Big) . \quad (5)$$

Note that if $\mathcal{H}$ is planar then, for all the leaves of $\tilde{\mathcal{T}}$, $\prod^{\rightarrow}_{\ell \neq l(f,\tilde{\mathcal{T}})} h_\ell^{\sigma^{\ell|f}}$ involves only lines belonging to $\mathcal{T}$, hence the relation corresponding to a leaf face implies $h_{l(f,\tilde{\mathcal{T}})} = e$ and, iterating, $h_l = e$, $\forall l \in \mathcal{H}$. Using eq. (5), eq. (4) writes

$$A^{\mathcal{G}} = \frac{(\lambda\bar{\lambda})^p}{[\delta^N(e)]^{p\frac{(D-2)(D-1)}{2}}} \int \prod_{\ell \in \mathcal{L}_{\mathcal{G}} \setminus \tilde{\mathcal{T}}} dh_\ell \prod_{l \in \tilde{\mathcal{T}}} d\tilde{h}_l \Big[\prod_{f' \notin \mathcal{H}} \delta^N_{f'}(\dots) \Big] \delta^N_r(\dots) \Big[\prod_{f \in \mathcal{H}, f \neq r} \delta^N_f \Big(\tilde{h}_{l(f,\tilde{\mathcal{T}})} \Big) \Big] , \quad (6)$$

where $\tilde{h}_{l(f,\tilde{\mathcal{T}})} = h_{l(f,\tilde{\mathcal{T}})}^{\sigma^{l(f,\tilde{\mathcal{T}})|f}} (\prod^{\rightarrow}_{\ell \neq l(f,\tilde{\mathcal{T}})} h_\ell^{\sigma^{\ell|f}})$. Integrating $\tilde{h}_{l(f,\tilde{\mathcal{T}})}$ and bounding the remaining delta functions by $\delta^N(e)$ we obtain a *jacket bound* [1]

$$A^{\mathcal{G}} \leq (\lambda\bar{\lambda})^p [\delta^N(e)]^{-p\frac{(D-2)(D-1)}{2} + \mathcal{F}_{\mathcal{G}} - \mathcal{F}_{\mathcal{H}} + 1} , \quad (7)$$

and $A^{\mathcal{G}}$ saturates eq. (7) if $\mathcal{H}$ is planar. The number of lines of $\mathcal{G}$ is $\mathcal{L}_{\mathcal{G}} = (D+1)p$ hence the number of faces of a jacket $\mathcal{J}$ (with genus $g_{\mathcal{J}}$) is $\mathcal{F}_{\mathcal{J}} = (D-1)p + 2 - 2g_{\mathcal{J}}$. Taking into account that $\mathcal{G}$ has $\frac{1}{2}D!$ jackets and each face belongs to $(D-1)!$ jackets,

$$\mathcal{F}_{\mathcal{G}} = \frac{D(D-1)}{2} p + D - \frac{2}{(D-1)!} \sum_{\mathcal{J}} g_{\mathcal{J}} , \quad (8)$$

and eq. (7) translates into

$$\begin{aligned} A^{\mathcal{G}} &\leq (\lambda\bar{\lambda})^p [\delta^N(e)]^{D-1-\frac{2}{(D-1)!}\sum_{\mathcal{J}} g_{\mathcal{J}} + 2g_{\mathcal{H}}} \\ &\leq (\lambda\bar{\lambda})^p [\delta^N(e)]^{D-1-\frac{2(D-2)}{D!}\sum_{\mathcal{J}} g_{\mathcal{J}}} , \end{aligned} \quad (9)$$

where for the last inequality we chose $\mathcal{H}$ with $g_{\mathcal{H}} = \inf_{\mathcal{J}} g_{\mathcal{J}}$. We conclude

- A graph $\mathcal{G}$ having *at least* a non planar jacket has amplitude bounded by $A^{\mathcal{G}} \leq (\lambda\bar{\lambda})^p [\delta^N(e)]^{D-1-\frac{2(D-2)}{D!}}$.
- A graph $\mathcal{G}$ whose *all* jackets are planar saturates the bound (9), $A^{\mathcal{G}} = (\lambda\bar{\lambda})^p [\delta^N(e)]^{D-1}$, and contributes to the leading order in $1/N$.

A first example of a graph whose all jackets are planar is obtained by reconnecting the two lines of color i in the graph $\mathcal{G}_1$ drawn in figure 2. We will denote this graph $\mathcal{S}$. Remark that $\mathcal{S}$ is dual to two D-simplices identified coherently along their $D-1$ boundary simplices i.e. the sphere S^D. Our result, eq. (1), is achieved by the following theorem.

Theorem 1. *If all the jackets $\mathcal{J}$ of a $D+1$ colored graph $\mathcal{G}$ are planar then the graph is dual to a sphere S^D.*

Proof: The theorem obviously holds for $D = 2$. Eq. (5) implies $h_l = e$, $\forall l \in \mathcal{G}$, hence the dual of $\mathcal{G}$ is simply connected. In $D = 3$ one concludes by the Poincaré-Perelman theorem that $\mathcal{G}$ is dual to a sphere. But in $D \geq 4$ one should check that *all* homotopy groups of the dual of $\mathcal{G}$ coincide with the ones of the sphere, which is cumbersome. Let's use instead induction on D.

Step 1: Contracting a full set of 1-Dipoles. As all the jackets $\mathcal{J}$ are planar, by lemma 1 all $\mathcal{J}^{\widehat{i}}_{(\rho)}$ are planar,

hence (by the induction hypothesis) all $\mathcal{B}^{\widehat{i}}_{(\rho)}$ are dual to spheres S^{D-1}.

Considering the D-bubbles $B^{\widehat{0}}_{(\rho)}$ as effective vertices one can choose a "connectivity tree" $\mathcal{T}^0$ of lines of color 0 (a set of lines connecting all $B^{\widehat{0}}_{(\rho)}$ without forming loops) [1]. All the lines in $\mathcal{T}^0$ are 1-Dipoles and can be contracted, hence $\mathcal{G}$ is equivalent with a graph with only one D-bubble $\mathcal{B}^{\widehat{0}}$. Any further contractions of 1-Dipoles of colors $1, 2, \dots D$ cannot disconnect $\mathcal{B}^{\widehat{0}}$. The genus of the jackets $\mathcal{J}$ does not change under 1-Dipole contractions (the number of vertices lines and faces of any jacket decreases by 2, $D+1$ and $D-1$ respectively).

Iterating for all colors, $\mathcal{G} \sim \mathcal{G}'$ with $\mathcal{G}'$ a graph having exactly one D-bubble $\mathcal{B}^{\widehat{i}}$, $\forall i$ and only planar jackets.

Step 2: $\mathcal{G}' = \mathcal{S}$. Consider first an arbitrary graph $\mathcal{G}$. Each of its bubbles $\mathcal{B}^{\widehat{i}}_{(\rho)}$ (with $\mathcal{N}_{\mathcal{B}^{\widehat{i}}_{(\rho)}} = 2p^{\widehat{i}}_{(\rho)}$) is a D-colored graph hence by eq. (8)

$$\mathcal{F}_{\mathcal{B}^{\widehat{i}}_{(\rho)}} = \frac{(D-1)(D-2)}{2} p^{\widehat{i}}_{(\rho)} + (D-1) - \frac{2}{(D-2)!} \sum_{\mathcal{J}^{\widehat{i}}_{(\rho)}} g_{\mathcal{J}^{\widehat{i}}_{(\rho)}} \ . \tag{10}$$

Each vertex of $\mathcal{G}$ contributes to $D+1$ of its D-bubbles ($\sum_{i;\rho} p^{\widehat{i}}_{\rho} = (D+1)p$), and each face to $D-1$ of them. Adding eq. (10) yields

$$\mathcal{F}_{\mathcal{G}} = \frac{(D-2)(D+1)}{2} p + \sum_{i;\rho} \Big(1 - \frac{2}{(D-1)!} \sum_{\mathcal{J}^{\widehat{i}}_{(\rho)}} g_{\mathcal{J}^{\widehat{i}}_{(\rho)}} \Big) \ , \tag{11}$$

which equated with (8) writes

$$\sum_{\mathcal{J}} g_{\mathcal{J}} = \frac{D!}{2} + \frac{(D-1)!}{2} p - \sum_{i;\rho} \Big(\frac{(D-1)!}{2} - \sum_{\mathcal{J}^{\widehat{i}}_{(\rho)}} g_{\mathcal{J}^{\widehat{i}}_{(\rho)}} \Big) \ . \tag{12}$$

For the graph $\mathcal{G}'$ we have $g_{\mathcal{J}} = g_{\mathcal{J}^{\widehat{i}}_{(\rho)}} = 0$ and, for all i, the sum over ρ has a single term. It follows that $\mathcal{G}'$ has exactly two vertices ($p = 1$) hence $\mathcal{G}' = \mathcal{S}$. Thus $\mathcal{G} \sim \mathcal{S}$ and the dual of $\mathcal{G}$ is homeomorphic to the sphere S^D.

□

In $D = 3$ we have $\mathcal{B}^{\widehat{i}}_{(\rho)} = \mathcal{J}^{\widehat{i}}_{(\rho)}$ and eq. (12) becomes $\sum_{\mathcal{J}} g_{\mathcal{J}} = 3 + p - \sum_{i;\rho} \Big(1 - g_{\mathcal{B}^{\widehat{i}}_{(\rho)}} \Big)$. The key to the full topological expansion of [1] lies in the fact that one can always cancel most planar bubbles by 1-Dipole moves. The final graphs obtained by this procedure (Core Graphs [1]) admit a bound in the number of vertices and index topologies.

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